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Variations on Quasi-Cauchy Sequences

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Abstract. In this paper, we introduce and study new kinds of continuities. It turns out that a function f defined on an interval is uniformly continuous if and only if there exists a positive integer p such that f preserves p-quasi-Cauchy sequences where a sequence (x_n) is called p-quasi-Cauchy if the sequence of differences between p-successive terms tends to 0.

1. Introduction

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([5, 24, 33]), quasi-slowly oscillating continuity ([29]), Δ-quasi-slowly oscillating continuity ([9]), ward continuity ([8], [2]), δ-ward continuity ([11]), statistical ward continuity, ([12]), λ-statistically ward continuity ([25]), N_θ-ward continuity ([15, 21]), ideal ward continuity ([16, 20]) and Abel continuity ([17]) which enabled some authors to obtain some characterizations of uniform continuity in terms of sequences in the sense that a function on a special subset of $\mathbb{R}$ preserves one of the following types of sequences: quasi-Cauchy sequences, statistical quasi-Cauchy sequences, λ-statistical quasi-Cauchy sequences, ideal quasi-Cauchy sequences, strongly lacunary quasi-Cauchy sequences, slowly oscillating sequences. A sequence (x_n) of points in $\mathbb{R}$ is slowly oscillating if $\lim_{\lambda\to 1^+} \overline{\lim}_n \max_{n+1\leq k\leq[\lambda n]} |x_k - x_n| = 0$ where $[\lambda n]$ denotes the integer part of λn ([28]). A sequence (x_n) is called statistically convergent to L ([30], [7], [3]) if $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$. Recently in [2] it was proved that a real valued function f defined on an interval is uniformly continuous if and only if it is ward continuous. What happens if we replace ”ward continuity” with ”2-ward continuity”. As a matter of fact we could replace ”ward continuity” with ”p-ward continuity” for any positive integer p.

The purpose of this paper is to introduce p-quasi-Cauchy sequences, and provide with interesting characterizations.

2. Variations on ward compactness

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers, and the set of positive integers. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in $\mathbb{R}$. p, c, and Δ will

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stand for a fixed element of $\mathbb{N}$, the set of all convergent sequences, and the set of all quasi-Cauchy sequences of points in $\mathbb{R}$, respectively. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero, and more generally speaking, than that the distance between p-successive terms is tending to zero where by p-successive terms we mean x_{n+p} and x_n. Nevertheless, sequences which satisfy this weaker property are interesting in their own right.

Definition 2.1. *A sequence (x_n) is called p-quasi-Cauchy if $\lim_{n\to\infty} \Delta_p x_n = 0$ where $\Delta_p x_n = x_{n+p} - x_n$ for every $n, p \in \mathbb{N}$.*

Note that $\mathbf{x}$ is quasi-Cauchy when $p = 1$, i.e. 1-quasi-Cauchy sequences are quasi-Cauchy sequences. We will denote the set of all p-quasi-Cauchy sequences by Δ_p for each $p \in \mathbb{N}$. It follows from the equality

$$x_{n+p} - x_n = x_{n+p} - x_{n+p-1} + x_{n+p-1} - +... - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n$$

that any quasi-Cauchy sequence is also p-quasi-Cauchy for any $p \in \mathbb{N}$, but the converse is not always true as it can be seen by considering the sequence $((-1)^n)$. Any slowly oscillating sequence is p-quasi-Cauchy for each $p \in \mathbb{N}$. On the other hand, neither, being δ-quasi-Cauchyness implies p-quasi-Cauchyness, nor p-quasi-Cauchyness implies δ-quasi-Cauchyness. Counterexamples for these situations are the sequences (n), and $((-1)^n)$, respectively. We note that the sum of two p-quasi-Cauchy sequence is p-quasi-Cauchy whereas the product of two p-quasi-Cauchy sequences need not be p-quasi-Cauchy as it can be seen by considering the product of the sequence $(\sqrt{n})$ itself. Cauchy sequences have the property that any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for p-quasi-Cauchy sequences. A counter example is the sequence $(\sqrt{n})$.

Definition 2.2. *A subset E of $\mathbb{R}$ is called p-ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a p-quasi-Cauchy subsequence.*

We note that this definition of p-ward compactness cannot be obtained by any summability matrix A, even by the summability matrix $A = (a_{kn})$ defined by $a_{kn} = -1$ if $k = n$, and $a_{kn} = 1$ if $k = n + p$; $a_{kn} = 0$ otherwise,

$$G(x) = \lim A\mathbf{x} = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn} a_n = \lim_{k\to\infty} \Delta_p x_k \qquad (*)$$

(see [6] for the definition of G-sequential compactness). In this case, despite that G-sequential compact subsets of $\mathbb{R}$ should include the singleton set $\{0\}$, p-ward compact subsets of $\mathbb{R}$ do not have to include the singleton $\{0\}$. Since any quasi-Cauchy sequence is p-quasi-Cauchy we see that any ward compact subset of $\mathbb{R}$ is p-ward compact for any $p \in \mathbb{N}$. We see that any finite subset of $\mathbb{R}$ is p-ward compact, the union of finite number of p-ward compact subsets of $\mathbb{R}$ is p-ward compact and the intersection of any family of p-ward compact subsets of $\mathbb{R}$ is p-ward compact. Furthermore any subset of a p-ward compact set of $\mathbb{R}$ is p-ward compact and any bounded subset of $\mathbb{R}$ is p-ward compact. These observations above suggest to us the following.

Theorem 2.3. *A subset E of $\mathbb{R}$ is bounded if and only if it is p-ward compact.*

Proof. It is easy to see that bounded subsets of $\mathbb{R}$ are p-ward compact. To prove the converse suppose that E is unbounded. If it is unbounded above, then one can construct a sequence (x_n) of numbers in E such that $x_{n+1} > p + x_n$ for each positive integer n. Then the sequence (x_n) does not have any p-quasi-Cauchy subsequence, so A is not p-ward compact. If E is bounded above and unbounded below, then similarly we obtain that E is not p-ward compact. This completes the proof. □

Corollary 2.4. A subset of $\mathbb{R}$ is ward compact if and only if it is p-ward compact for a positive integer p.

Proof. Using the fact that totally boundedness coincides with boundedness in $\mathbb{R}$, the proof follows from Theorem 3 of [10], and Theorem 2.3, so is omitted. □

Corollary 2.5. A subset of $\mathbb{R}$ is statistically ward compact if and only if it is p-ward compact for a positive integer p.

Proof. The proof follows from Lemma 2 of [12] and Theorem 2.3, so is omitted. □

Corollary 2.6. A subset of $\mathbb{R}$ is strongly lacunary ward compact if and only if it is p-ward compact for a positive integer p.

Proof. The proof follows from Theorem 3.3 of [15] and Theorem 2.3, so is omitted. □

Corollary 2.7. A subset of $\mathbb{R}$ is slowly oscillating compact if and only if it is p-ward compact for a positive integer p.

Proof. The proof follows from Theorem 3 in [10], and Theorem 2.3, so is omitted. □

Corollary 2.8. If a closed subset of $\mathbb{R}$ is p-ward compact for a positive integer p, then any sequence $\mathbf{x} = (x_n)$ of points in E has a (P_n, s)-absolutely almost convergent subsequence (see [26], [1], and [18] for the related definitions).

3. Variations on ward continuity

There are interesting connections between uniformly continuous functions and p-quasi-Cauchy sequences. We mainly investigate these connections in this section. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves convergent sequences. Using this idea of sequential continuity of a real function in terms of sequences, we introduce p-ward continuity.

Definition 3.1. *A function f is called p-ward continuous on E if f preserves p-quasi-Cauchy sequences, i.e. the sequence $f(\boldsymbol{x}) = (f(x_n))$ is p-quasi-Cauchy whenever $\mathbf{x} = (x_n)$ is a sequence of terms in E which is p-quasi-Cauchy.*

We note that this definition of p-ward continuity can not be obtained by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by (∗) however for this special summability matrix A if A-continuity of f at the point 0 implies p-ward continuity of f, then $f(0) = 0$; and if p-quasi-Cauchy continuity of f implies A-continuity of f at the point 0, then $f(0) = 0$ (see [4]).

We also note that the sum of two p-ward continuous functions is p-ward continuous, and the the composition of two p-ward continuous functions is p-ward continuous, but the product of two p-ward continuous functions need not be p-ward continuous as it can be seen by considering product of the p-ward continuous function $f(x) = x$ with itself.

In connection with p-quasi-Cauchy sequences, slowly oscillating sequences, and convergent sequences the problem arises to investigate the following types of continuity of a function on $\mathbb{R}$.

(Δ_p) $(x_n) \in \Delta_p \Rightarrow (f(x_n)) \in \Delta_p$

($\Delta_p c$) $(x_n) \in \Delta_p \Rightarrow (f(x_n)) \in c$

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$

(d) $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta_p$

(e) $(x_n) \in w \Rightarrow (f(x_n)) \in \Delta_p$

where w denotes the set of slowly oscillating sequences. We see that (Δ_p) is p-ward continuity of f, and (c) states the ordinary continuity of f. It is easy to see that $(\Delta_p c)$ implies (Δ_p), and (Δ_p) does not imply $(\Delta_p c)$; and (Δ_p) implies (d), and (d) does not imply (Δ_p); and (Δ_p) implies (e), and (e) does not imply (Δ_p); $(\Delta_p c)$ implies (c) and (c) does not imply $(\Delta_p c)$; and (c) is equivalent to (d). There is no p-ward continuous function which satisfies that $(x_n) \in c \Rightarrow (f(x_n)) \in c_0$ where c_0 denotes the set of all null sequences. We note that if $(x_n) \in c_0$ implies that $(f(x_n)) \in c$, then $\lim_{n\to\infty} f(x_n) = f(0)$. Now we give the implication (Δ_p) implies (Δ_1), i.e. any p-ward continuous function is 1-ward continuous, i.e. ward continuous.

Theorem 3.2. *If f is p-ward continuous on a subset E of $\mathbb{R}$ for a $p \in \mathbb{N}$, then it is ward continuous on E.*

Proof. If $p = 1$, then there is nothing to prove. So we would suppose that $p > 1$. Take any p-ward continuous function f on E. Let (x_n) be any quasi-Cauchy sequence of points in E. Then the sequence

$$(x_1, x_1, ..., x_1, x_2, x_2, ..., x_2, ..., x_n, x_n, ..., x_n, ...)$$

is also quasi-Cauchy so it is p-quasi-Cauchy, hence it belongs to Δ_p so does the sequence $(f(x_1), f(x_1), ..., f(x_1), f(x_2), f(x_2), ..., f(x_2), ..., f(x_n), f(x_n), ..., f(x_n), ...)$ where the same term repeats p-times. Thus $\lim_{n\to\infty}(f(x_{n+p}) - f(x_n)) = 0$. This completes the proof of the theorem. □

Corollary 3.3. *If f is p-ward continuous on a subset E of $\mathbb{R}$, then it is continuous on E in the ordinary case.*

Proof. The proof follows immediately from Theorem 1 on page 228 in [8] so is omitted. □

Theorem 3.4. *p-ward continuous image of any p-ward compact subset of $\mathbb{R}$ is p-ward compact.*

Proof. Let f be a p-ward continuous function and E be a p-ward compact subset of $\mathbb{R}$. Take any sequence $\mathbf{y} = (y_n)$ of terms in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbb{N}$. p-ward compactness of E implies that there is a p-quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$. Since f is p-ward continuous, $(t_k) = f(\mathbf{z}) = (f(z_k))$ is p-quasi-Cauchy. Thus (t_k) is a p-quasi-Cauchy subsequence of the sequence $f(\mathbf{x})$. This completes the proof of the theorem. □

Corollary 3.5. *p-ward continuous image of any compact subset of $\mathbb{R}$ is compact.*

Proof. The proof follows from the preceding theorem, so is omitted. □

Corollary 3.6. *p-ward continuous image of any G-sequentially connected subset of $\mathbb{R}$ is G-sequentially connected for a regular subsequential method G.*

Proof. The proof follows from the preceding theorem, so is omitted (see [14] for the definition of G-sequential connectedness). □

Now we prove that any uniformly continuous function preserves p-quasi-Cauchy sequences.

Theorem 3.7. *If f is uniformly continuous on a subset E of $\mathbb{R}$, then it is p-ward continuous on E for any $p \in \mathbb{N}$.*

Proof. Let f be uniformly continuous function E. To prove that $(f(x_n))$ is a p-quasi-Cauchy sequence whenever (x_n) is, take any $\varepsilon > 0$. Uniform continuity of f on E implies that there exists a $\delta > 0$, depending on ε, such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in E$. For this $\delta > 0$, there exists an $N = N(\delta) = N_1(\varepsilon)$ such that $|\Delta_p x_n| < \delta$ whenever $n > N$. Hence $|\Delta_p f(x_n)| < \varepsilon$ if $n > N$. It follows from this that $(f(x_n))$ is a p-quasi-Cauchy sequence. This completes the proof of the theorem. □

Corollary 3.8. *If f is slowly oscillating continuous on an interval E, then it is p-ward continuous for all $p \in \mathbb{N}$.*

Proof. If f is a slowly oscillating continuous function on an interval E, then it is uniformly continuous on E by Theorem 5 on page 1623 in [10]. Hence it follows from the preceding theorem that f is p-ward continuous on E for all $p \in \mathbb{N}$. □

It is well-known that any continuous function on a compact subset E of $\mathbb{R}$ is uniformly continuous on E. We have an analogous theorem for a p-ward continuous function defined on a p-ward compact subset of $\mathbb{R}$.

Theorem 3.9. *If a function is p-ward continuous on a p-ward compact subset E of $\mathbb{R}$, then it is uniformly continuous on E.*

Proof. Suppose that f is not uniformly continuous on E so that there exist an $\epsilon_0 > 0$ and sequences (x_n) and (y_n) of points in E such that

$$|x_n - y_n| < 1/n$$

and

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbb{N}$. Since E is p-ward compact, there is a subsequence (x_{n_k}) of (x_n) that is p-quasi-Cauchy. On the other hand there is a subsequence $(y_{n_{k_j}})$ of (y_{n_k}) that is p-quasi-Cauchy as well. It is clear that the corresponding sequence $(x_{n_{k_j}})$ is also p-quasi-Cauchy, since $(y_{n_{k_j}})$ is p-quasi-Cauchy and

$$|x_{n_{k_j}} - x_{n_{k_{j+p}}}| \leq |x_{n_{k_j}} - y_{n_{k_j}}| + |y_{n_{k_j}} - x_{n_{k_{j+p}}}| + |x_{n_{k_{j+p}}} - x_{n_{k_{j+p}}}|.$$

which follows from the inequality

$$|y_{n_{k_{j+p}}} - x_{n_{k_{j+p}}}| \leq |y_{n_{k_{j+p}}} - y_{n_{k_j}}| + |y_{n_{k_j}} - x_{n_{k_j}}| + |x_{n_{k_j}} - x_{n_{k_{j+p}}}|.$$

Now the sequence

$$(x_{n_{k_1}}, x_{n_{k_1}}, ..., x_{n_{k_1}}, y_{n_{k_1}}, y_{n_{k_1}}, ..., y_{n_{k_1}}, ..., x_{n_{k_j}}, x_{n_{k_j}}, ..., x_{n_{k_j}}, y_{n_{k_j}}, y_{n_{k_j}}, ..., y_{n_{k_j}}, ...)$$

is p-quasi-Cauchy while the transformed sequence is not p-quasi-Cauchy where same terms repeat p-times. Hence this establishes a contradiction so completes the proof of the theorem. □

Combining Theorem 2.3, Theorem 3.7, and the preceding theorem we have the following:

Corollary 3.10. *A function defined on a bounded subset of $\mathbb{R}$ is uniformly continuous if and only if it is p-ward continuous.*

When the domain of a function is restricted to a bounded subset of $\mathbb{R}$, p-ward continuity coincides with not only ward continuity, but also any one of the following continuities: slowly oscillating continuity, statistically ward continuity, λ-statistically ward continuity, ideal ward continuity, and N_θ-ward continuity. We see that for a regular subsequential method G that any p-ward continuous function on a G-sequentially compact or a G-sequentially connected subset E of $\mathbb{R}$ is uniformly continuous on E (see [13], [14], [22]).

Lemma 3.11. *If (ξ_n, η_n) is a sequence of ordered pairs of points in an interval E of $\mathbb{R}$ such that $\lim_{n\to\infty} |\xi_n - \eta_n| = 0$, then there exists a p-quasi-Cauchy sequence (x_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-p}, x_j)$.*

Proof. For each positive integer k, we can fix $y_0^k, y_1^k, ..., y_{n_k}^k$ in E with $y_0^k = \eta_k$, $y_{n_k}^k = \xi_{k+1}$, and $|y_i^k - y_{i-p}^k| < \frac{1}{k}$ for $1 \leq i \leq n_k$. Now write

$$(\xi_1, \eta_1, y_1^1, ..., y_{n_1-1}^1, \xi_2, \eta_2, y_1^2, ..., y_{n_2-1}^2, \xi_3, \eta_3, ..., \xi_k, \eta_k, y_1^k, ..., y_{n_{k-1}}^k, \xi_{k+1}, \eta_{k+1}, ...)$$

Then denoting this sequence by (x_n) we obtain that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-p}, x_j)$. This completes the proof of the lemma. □

Theorem 3.12. *If f is is p-ward continuous on an interval E for a positive integer p, then it is uniformly continuous on E.*

Proof. Let p be any positive integer and E an interval. To prove that p-ward continuity of f on E implies uniform continuity of f on E suppose that f is not uniformly continuous on E so that there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there exist $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Hence for each positive integer n, there exist x_n and y_n in E such that $|x_n - y_n) < \frac{1}{n}$, and $|f(x_n) - f(y_n)| \geq \varepsilon$. By Lemma 3.11, one can construct a p-quasi-Cauchy sequence (t_n) which has a subsequence $(z_n) = (t_{k_n})$ such that $\lim_{n\to\infty}(z_{n+p} - z_n) = 0$, but $|f(z_{n+p}) - f(z_n)| \geq \varepsilon$. Therefore the transformed sequence $(f(z_n))$ is not p-quasi-Cauchy. Thus this contradiction yields that p-ward continuity implies uniform continuity. This completes the proof of the theorem. □

Combining Theorem 3.7 and the preceding theorem we have the following:

Corollary 3.13. *A function f is uniformly continuous on an interval E if and only if it is p-ward continuous for a positive integer p.*

Corollary 3.14. *A function defined on an interval is p-ward continuous if and only if it is slowly oscillating continuous.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in the case of p-ward continuity; i.e. uniform limit of a sequence of p-ward continuous functions is p-ward continuous.

Theorem 3.15. *If (f_n) is a sequence of p-ward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is p-ward continuous on E.*

Proof. Let $\mathbf{x} = (x_n)$ be any p-quasi-Cauchy sequence of points in E and $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is p-ward continuous, there exists a positive integer N_1, depending on ε and greater than N such that $|f_N(x_{n+p}) - f_N(x_n)| < \frac{\varepsilon}{3}$ for $n \geq N_1$. Now for $n \geq N_1$ we have
$|f(x_{n+p}) - f(x_n)| \leq |f(x_{n+p}) - f_N(x_{n+p})| + |f_N(x_{n+p}) - f_N(x_n)| + |f_N(x_n) - f(x_n)| \quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$
This completes the proof of the theorem. □

Theorem 3.16. *The set of all p-ward continuous functions defined on a subset E of $\mathbb{R}$ is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta_p FC(E)} = \Delta_p FC(E)$ where $\Delta_p FC(E)$ is the set of all p-ward continuous functions on E, $\overline{\Delta_p FC(E)}$ denotes the set of all cluster points of $\Delta_p FC(E)$.*

Proof. Let f be any element in the closure of $\Delta_p FC(E)$. Then there exists a sequence of points in $\Delta_p FC(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is p-ward continuous, take any p-quasi-Cauchy sequence (x_n) of points in E. Let $\varepsilon > 0$. Since (f_k) converges to f, there exists an N such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$. As f_N is p-ward continuous, there is an N_1, greater than N, such that for all $n \geq N_1$, $|f_N(x_{n+p}) - f_N(x_n)| < \frac{\varepsilon}{3}$. Hence for all $n \geq N_1$,
$|f(x_{n+p}) - f(x_n)| \leq |f(x_{n+p}) - f_N(x_{n+p})| + |f(x_n) - f_N(x_n)| + |f_N(x_{n+p}) - f_N(x_n)| \quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$
This completes the proof of the theorem. □

Corollary 3.17. *The set of all p-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E for a fixed positive integer p.*

Proof. The proof follows from the preceding theorem. □

4. Conclusion

In this paper, we obtain some characterizations of uniform continuity introducing a concept of p-ward continuity of a real function and a concept of p-ward compactness of a subset of $\mathbb{R}$ via sequences which satisfy that p-th forward difference sequence tends to 0, and investigate some other results about this kind of continuity and some other kinds of compactness. It turns out that p-ward continuity coincides with uniform continuity not only on an interval, but also on a bounded subset of $\mathbb{R}$. For the special case $p = 1$ one can obtain the results in [8], [2], and [12]. p-quasi Cauchy concept for $p > 1$ might find more interesting applications than quasi Cauchy sequences find to the cases when 1-forward difference does not apply. For a further study, we suggest to investigate p-quasi-Cauchy sequences of fuzzy points and p-ward continuity for the fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [19], and [31]). We also suggest to investigate p-quasi-Cauchy double sequences of points in $\mathbb{R}$ (see [32], [27], and [23] for the related definitions in the double case).

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